Dimensional Regularization and Nuclear Potentials

J.L. Friar
Theoretical Division
Los Alamos National Laboratory
Los Alamos, NM 87545

Abstract

It is shown how nucleon-nucleon potentials can be defined in N dimensions, using dimensional regularization to continue amplitudes. This provides an easy way to separate out contact (δ-function) terms arising from renormalization. An example is worked out several ways for the case of two scalar particles exchanged between nucleons, which involves a very simple loop calculation. This leads to a Feynman-parameterized representation for the nucleon-nucleon potential. Alternately, a dispersion representation can be developed leading to a different, though equivalent, form.

It might be surmised that there exist no **new** ways to develop nuclear potentials after decades of experimentation with techniques. We present below a variation on several such techniques, which, to the best of our knowledge, has not been used before. As an illustration, we perform in several different ways a simple calculation involving a closed meson loop, which leads to a two-boson-exchange potential (TBEP).

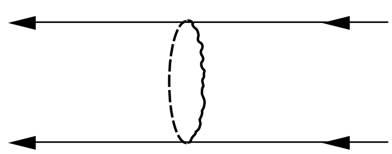

Figure 1: The nucleon-nucleon potential resulting from the simultaneous exchange of two scalar particles, "a" and "b".

Figure 1 shows the interaction of two nucleons "1" and "2" via the exchange of two different scalar mesons, "a" and "b". The corresponding Lagrangian is

$$L_{INT} = \lambda \overline{N} N a b \, . \tag{1}$$

The nonrelativistic potential corresponding to Figure (1) can be developed in old-fashioned second-order perturbation theory in the usual way[1] for a nucleon-nucleon separation, r:

$$V_{12}(r) = -2\lambda^2 \int \frac{d^3 q_a}{(2\pi)^3} \frac{e^{i\boldsymbol{q}_a \cdot \boldsymbol{r}}}{(2E_a)} \int \frac{d^3 q_b}{(2\pi)^3} \frac{e^{i\boldsymbol{q}_b \cdot \boldsymbol{r}}}{(2E_b)(E_a + E_b)} . \tag{2}$$

This awkward and inelegant expression is nevertheless very easy to interpret. The initial factor of two accounts for two ways the mesons can be exchanged: “1” to “2” or from “2” to “1”. The factors $(2E_a)$ and $(2E_b)$ are the wave function normalization factors for “a” and “b” ($E_x = (q_x^2 + m_x^2)^{\frac{1}{2}}$), while $-(E_a + E_b)$ is the energy denominator (neglecting nucleon recoil) and the integration is over the mesons' phase spaces. This can be cast into a more conventional form by using[2]

$$\frac{1}{E_a E_b (E_a + E_b)} = \frac{2}{\pi} \int_0^\infty \frac{d\beta}{(E_a^2 + \beta^2)(E_b^2 + \beta^2)} , \tag{3}$$

which allows a convolution representation to be written in terms of separate Yukawa functions for a and b exchange:

$$V_{12}(r) = \frac{-4\lambda^2}{(4\pi)^3 r^2} \int_0^\infty d\beta\, e^{-\left(\sqrt{\beta^2 + m_a^2} + \sqrt{\beta^2 + m_b^2}\right) r} \tag{4a}$$

$$= \frac{-2\lambda^2}{(4\pi)^3 r^2} \int_{m_a + m_b}^\infty \frac{dy\, e^{-yr}}{y^2} \frac{(y^2 + m_a^2 - m_b^2)(y^2 + m_b^2 - m_a^2)}{[y^4 - 2y^2(m_a^2 + m_b^2) + (m_a^2 - m_b^2)^2]^{\frac{1}{2}}} \tag{4b}$$

$$= \frac{-2\lambda^2}{(4\pi)^3 r} \int_{m_a + m_b}^\infty \frac{dy}{y} e^{-yr} [y^4 - 2y^2(m_a^2 + m_b^2) + (m_a^2 - m_b^2)^2]^{\frac{1}{2}} . \tag{4c}$$

We have made a change of variables to obtain eq. (4b) and an integration by parts to realize eq. (4c). Our points can be made if we further assume that one particle mass vanishes or that $m_a = m_b = m$; the latter leads to an elementary integral which we evaluate:

$$V_{12}(r) = \frac{-2\lambda^2}{(4\pi)^3 r^2} \int_{2m}^\infty dy\, e^{-yr} \frac{y}{(y^2 - 4m^2)^{\frac{1}{2}}} \tag{5a}$$

$$= \frac{-4m\lambda^2}{(4\pi)^3 r^2} K_1(2mr) . \tag{5b}$$

Note the factor of $(4\pi)^3$, one power from the configuration space propagator and two powers from the loop integral. Both are necessary for dimensional power counting to work[3].

On the other hand, we can calculate the Feynman diagram associated with the loop in Figure (1). It is easy to show[4] that the potential equivalent to that amplitude, M, should have an additional factor of i: $V = iM$. Evaluating the diagram we find

$$M(q) = (\overline{N}_1 N_1)(\overline{N}_2 N_2)\lambda^2 I(q^2) , \tag{6a}$$

where q is the transferred momentum. The divergent loop integral is given by

$$I(q^2) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - m_a^2)((q-k)^2 - m_b^2)}\,, \tag{6b}$$

and dropping the unimportant nucleon spinor factors we find

$$V(q^2) = i\lambda^2 I(q^2)\,. \tag{6c}$$

Evaluating divergent loop integrals requires regularization, and for a variety of reasons, dimensional regularization is the method of choice[5] today. This yields

$$I_N(q^2) \to \mu^{4-N} \int \frac{d^N k}{(2\pi)^N} \frac{1}{(k^2 - m_a^2)} \frac{}{((q-k)^2 - m_b^2)}\,, \tag{6d}$$

where the number of dimensions has been extended from 4 to N, and for N $<$ 4 the integral is convergent. The renormalization scale μ keeps the overall dimensionality of the integral the same.

Using the Feynman parameterization[4]

$$\frac{1}{ab} = \int_0^1 \frac{dz}{[a + (b-a)z]^2}\,, \tag{7}$$

and shifting the variable to $k' = k - qz$ leads to

$$I_N(q^2) = \int_0^1 dz \int \frac{d^N k'}{(2\pi)^N} \frac{1}{[k'^2 - (m_a^2 - (m_a^2 - m_b^2)z - q^2 z(1-z))]^2} \tag{8a}$$

$$= \frac{i}{(4\pi)^2}(4\pi\mu^2)^{2-\frac{N}{2}}\,\Gamma(2 - \frac{N}{2}) \int_0^1 dz\,[m_a^2 - (m_a^2 - m_b^2)z - q^2 z(1-z)]^{\frac{N}{2}-2}\,. \tag{8b}$$

The latter result is a standard form for one-loop amplitudes[6]. Usually at this point one writes $4 - N = \epsilon$ and performs an ϵ-expansion, leading to a divergent constant (which generates a contact term, $(\overline{N}N)^2$, in the nucleon-nucleon force) and a finite logarithm. We eschew this approach and define a potential in N space-time or n space dimensions ($n = N - 1$):

$$V_N(r) \equiv \int \frac{d^n q}{(2\pi)^n} e^{i\boldsymbol{q}\cdot\boldsymbol{r}} [i\lambda^2 I_N(q^2)]\,, \tag{9}$$

where we choose to work in the frame where $q_0 = 0$ (I_N is a function of $q^2 = q_0^2 - \boldsymbol{q}^2$). Alternatively, the energy transfer, q_0, is small and of order $(\frac{v}{c})^2$ (i.e., a relativistic correction) and can be dropped. Both $\boldsymbol{q}$ and $\boldsymbol{r}$ are vectors in n space dimensions. The angular integrals can be evaluated[5] which leads to

$$V_N(r) = \frac{r^{1-\frac{n}{2}}}{(2\pi)^{\frac{n}{2}}} \int_0^\infty dq\, q^{\frac{n}{2}} J_{\frac{n}{2}-1}(qr) \left[i\lambda^2 I_N(-q^2)\right]\,, \tag{10}$$

which can be verified easily for $n = 3$. Note that we are using $q^2 \equiv \boldsymbol{q}^2$ as an integration variable, which accounts for the sign change in the argument of I_N. Inserting expression (8b) for I_N and performing the q-integral leads to:

$$V_N(r) = \frac{-(4\pi\mu^2)^{2-\frac{N}{2}}\,\lambda^2}{\sqrt{2}(4\pi)^2\pi^{\frac{n}{2}}\,r^{n-\frac{3}{2}}}\int_0^1 \frac{dz\,\beta^{n-\frac{3}{2}}\,K_{n-\frac{3}{2}}(\beta r)}{[z(1-z)]^{2-\frac{N}{2}}}\,, \tag{11a}$$

where

$$\beta^2(z) = \frac{m_a^2 - (m_a^2 - m_b^2)z}{z(1-z)} > 0\,. \tag{11b}$$

The factors of 2 and π clearly depend on n (or N). Note that the divergent (for $N = 4$) Γ-function has disappeared and the result is finite for $r \neq 0$. Standard Bessel function identities[7] allow us to write for $N = 4$:

$$V_4 = \frac{-2\lambda^2}{(4\pi)^3 r^3}\int_0^1 dz\,e^{-\beta r}(1+\beta r)\,. \tag{12}$$

For $m_a = m_b$ this can be shown[8] to equal eq. (5b). Note that this representation for the potential involves the Feynman parameterization variable, z. Our derivation (12) is both elegant and more directly related to the Feynman loop diagrams than is the conventional derivation. Indeed, dimensional regularization was developed for these diagrams[5]

The reason why the divergent factor $\Gamma(2-\frac{N}{2})$ disappears is that we implicitly renormalized the loop graph when we developed the potential. Performing an ϵ-expansion ($\Gamma(2-\frac{N}{2}) \sim \frac{2}{\epsilon}+$ finite) in eq. (8b) leads to a contact term ($\sim \delta^3(\boldsymbol{r})$). By keeping the internucleon separation, r, finite, these terms don't arise. They have been "regularized" away. Note that $V_N(r) \sim \frac{1}{r^{2N-5}}$ for small r and is progressively more singular for increasing N.

We can derive this result and an alternative representation (equivalent to eq. (4c), for $N = 4$) by using the analytic properties of the field-theoretic amplitude. In the t-channel of Fig. (1) the reaction $\overline{N} + N \to a + b$ is open, and this information allows construction of a potential, an old but very useful technique[9]. The amplitude $I_N(q^2)$ in eq. (9) is real for $q^2 < 0$, but for $q^2 > 0$ it develops an imaginary part above the threshold for the reaction, which requires $q^2 \geq (m_a + m_b)$. Switching to the integration variable $q^2 \equiv \boldsymbol{q}^2$ used in eq. (10), this implies branch cuts for $q^2 < 0$ or along the imaginary axis in the complex q-plane indicated in Fig. (2). The left-hand cut arises from $q^{\frac{n}{2}} J_{\frac{n}{2}-1}(qx)$. Writing $J_\nu(z) = \frac{1}{2}\left[H_\nu^{(1)}(z) + H_\nu^{(2)}(z)\right]$ and noting that $H_\nu^{(1)}$ behaves asymptotically as e^{iz} while $H_\nu^{(2)}$ behaves as e^{-iz}, we can write

$$J_N = \int_0^\infty dq\,q^{\frac{n}{2}} J_{\frac{n}{2}-1}(qr) I_N(-q^2) = \frac{1}{2}\int_C dq\,q^{\frac{n}{2}} H^{(1)}_{\frac{n}{2}-1}(qr) I_N(-q^2)\,, \tag{13}$$

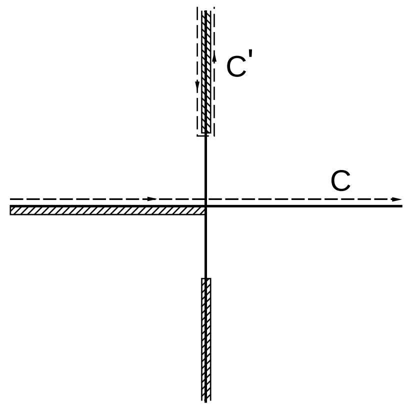

Figure 2: The analytic q-plane illustrating branch cuts discussed in the text. The integration contour-C, which defines the Fourier transform, can be deformed to C′, which gives the dispersion representation for the potential.

using $H_\nu^{(1)}(e^{i\pi}r) = H_\nu^{(2)}(r)e^{-\pi i(\nu+1)}$. The integral along C can be continuously deformed into C' because the integral vanishes exponentially in the upper half of the q-plane. Using the properties of the $H_\nu^{(1)}$ function for imaginary argument and $q = ix \pm \epsilon$, we then find the simple and elegant result:

$$J_N = \frac{2}{\pi} \int_{m_a+m_b}^{\infty} dx\, x^{\frac{n}{2}}\, K_{\frac{n}{2}-1}(xr)\, \Im \left[I_N(x^2 + i\epsilon) \right] \ , \tag{14}$$

which provides an excellent representation for the potential. Note that the range of the force varies from $(m_a + m_b)$ to ∞. Putting everything together from eqns. (8b) and (10), we obtain

$$V_N = \frac{-2\lambda^2 (4\pi\mu^2)^{2-\frac{N}{2}}\, \Gamma(2 - \frac{N}{2})}{\pi (2\pi)^{\frac{n}{2}} r^{\frac{n}{2}-1} (4\pi)^2} \int_{m_a+m_b}^{\infty} dx\, x^{\frac{n}{2}}\, K_{\frac{n}{2}-1}(xr)$$
$$\Im \left[\int_0^1 \frac{dz}{[z(1-z)]^{2-\frac{N}{2}} [\beta^2 - x^2 - i\epsilon]^{2-\frac{N}{2}}} \right] \ , \tag{15a}$$

where $\beta^2 = [m_a^2 - (m_a^2 - m_b^2)z]/z(1-z)$. Evaluating the imaginary part leads to

$$V_N = \frac{-2\lambda^2 (4\pi\mu^2)^{2-\frac{N}{2}}}{(2\pi)^{\frac{n}{2}} (4\pi)^2\, \Gamma(\frac{N}{2} - 1)\, r^{\frac{n}{2}-1}} \int_{m_a+m_b}^{\infty} dx\, x^{\frac{n}{2}}\, K_{\frac{n}{2}-1}(xr)$$
$$\int_0^1 \frac{dz\, \theta(x^2 - \beta^2)}{[z(1-z)(x^2 - \beta^2)]^{2-\frac{N}{2}}} \ , \tag{15b}$$

where the singularity at $N = 4$ has disappeared.

Two options are available for further simplification. One can first evaluate the x integral, which leads immediately to eq. (11a). Alternatively, one can perform the z integral first. The function $\beta(z)$ is larger than x^2 for values of z near 0 and 1, making the θ-function vanish. The argument of that function is positive between two values of z $\left(z_\pm = \frac{x^2+\Delta m^2}{2x^2} \pm \frac{1}{2}\left[(1+ \frac{\Delta m^2}{x^2})^2 - \frac{4m_a^2}{x^2}\right]^{\frac{1}{2}}\right)$, where $\Delta m^2 = m_a^2 - m_b^2$, which resets the limits on the integral. The resulting integral is a beta-function [7], and one obtains

$$V_N(r) = \frac{-2\lambda^2(4\pi\mu^2)^{2-N/2}\, 2^{3/2-n}}{(2\pi)^{\frac{n-1}{2}}\, r^{\frac{n}{2}-1}\, (4\pi)^2\, \Gamma(\frac{n}{2})} \int_{m_a+m_b}^{\infty} \frac{dx}{\sqrt{x}}\, K_{\frac{n}{2}-1}(xr) \left[(x^2+\Delta m^2)^2 - 4m_a^2 x^2\right]^{\frac{N-2}{4}} . \tag{16}$$

This reduces to eq. (4c) for $N = 4$.

Equations (11) and (16) are our principal results and are equivalent. The former is a potential defined in N dimensions which exploits the Feynman parameterization of the loop integral. The latter is a dispersion representation which exploits the analytic properties of the amplitude, and was also derived using a trick and simple, old-fashioned perturbation theory (in 4 dimensions). Infinities which naturally arise are regularized and disappear from the final form. Dimensional regularization was used to define the continuation to N dimensions, and both our approach and our techniques maintain a strong connection between the potential and the underlying field theory.

Acknowledgements

This work was performed under the auspices of the U. S. Department of Energy.

References

[1] J. L. Friar and S. A. Coon, Phys. Rev. C **49**, 1272 (1994).

[2] Th. A. Rijken, Ann. Phys. (N. Y.) **208**, 253 (1991).

[3] H. Georgi, *Weak Interactions and Modern Particle Theory*, (Benjamin, Menlo Park, 1984); A. Manohar and H. Georgi, Nucl. Phys. **B234**, 189 (1984).

[4] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics*, (McGraw-Hill, New York, 1964). We use the metric and conventions of this reference.

[5] G. 't Hooft and M. Veltman, *Diagrammar*, CERN Yellow Report 73-9.

[6] J. F. Donoghue, E. Golowich, and B. R. Holstein, *Dynamics of the Standard Model*, (University Press, Cambridge, 1992).

[7] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions*, (Dover, New York, 1965).

[8] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, ed. by A. Jeffrey, (Academic Press, Boston, 1994).

[9] G. E. Brown and A. D. Jackson, *The Nucleon-Nucleon Interaction*, (NORDITA,Copenhagen,1974), part II.